CSIDH:
An Efficient Post-Quantum Commutative Group Action

https://csidh.isogeny.org

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History

1976 Diffie-Hellman: Key exchange using exponentiation in groups (DH)

1985 Koblitz-Miller: Diffie-Hellman style key exchange using multiplication in elliptic curve groups (ECDH)

1990 Brassard-Yung: Generalizes ‘group exponentiation’ to ‘groups acting on sets’ in a crypto context

1994 Shor: Polynomial-time quantum algorithm to break the discrete logarithm problem in any group, quantumly breaking DH and ECDH

1997 Couveignes: Post-quantum isogeny-based Diffie-Hellman-style key exchange using commutative group actions (not published at the time)

2003 Kuperberg: Subexponential-time quantum algorithm to attack cryptosystems based on a hidden shift

History

2004 Stolbunov-Rostovtsev independently rediscover Couveignes' scheme (CRS)

2006 Charles-Goren-Lauter: Build hash function from supersingular isogeny graph

2010 Childs-Jao-Soukharev: Apply Kuperberg's (and Regev's) hidden shift subexponential quantum algorithm to CRS

2011 Jao-De Feo: Build Diffie-Hellman style key exchange from supersingular isogeny graph (SIDH)

2018 De Feo-Kieffer-Smith: Apply new ideas to speed up CRS

2018 Castryck-Lange-Martindale-Panny-Renes: Apply ideas of De Feo, Kieffer, Smith to supersingular curves over $\mathbb{F}_p$ (CSIDH)

(History slides mostly stolen from Wouter Castryck)

Why CSIDH?

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- Drop-in post-quantum replacement for (EC)DH
- Non-interactive key exchange (full public-key validation); previously an open problem post-quantumly
- Small keys: 64 bytes at conjectured AES-128 security level
- Competitive speed: $\sim 85\,\mathrm{ms}$ for a full key exchange
- Flexible:
  - Compatible with 0-RTT protocols such as QUIC
  - [DG] uses CSIDH for 'SeaSign' signatures
  - [DGOPS] uses CSIDH for oblivious transfer
  - [FTY] uses CSIDH for authenticated group key exchange

CSIDH vs SIDH?

Apart from mathematical background, SIDH and CSIDH actually have very little in common, and are likely to be useful for different applications.

Here is a comparison (mostly stolen from Luca de Feo):

	CSIDH	SIDH
Speed (NIST 1)	85ms	$\approx$ 10ms[1]
Public key size (NIST 1)	64B	378B
Key compression (speed)		$\approx$ 15ms
Key compression (size)		222B
Constant time implementation	yes (quick and dirty)	yes
Submitted to NIST	no	yes
Maturity	7 months	7 years
Best classical attack	$p^{1/4}$	$p^{1/4}$
Best quantum attack	subexponential	$p^{1/6}$
Key size scales	quadratically	linearly
Security assumption	isogeny walk problem	ad hoc
CPA security	yes	yes
CCA security	yes	Fujisaki-Okamoto
Non-interactive key exchange	yes	unbearably slow
Signatures (classical)	unbearably slow	seconds
Signatures (quantum)	seconds	still seconds?

[1] This is a very conservative estimate!

Post-quantum Diffie-Hellman?

Traditionally, Diffie-Hellman works in a group G via the map

$$\begin{array}{rcl} \mathbb{Z} \times G & \to & G \\ (x, g) & \mapsto & g^x. \end{array}$$

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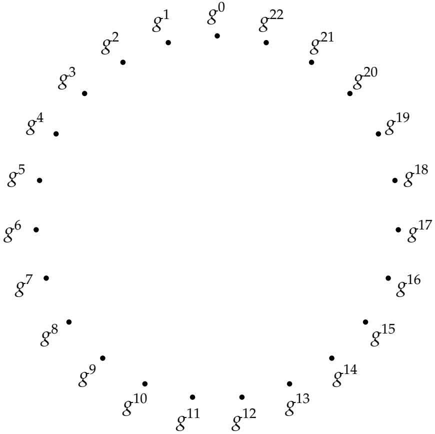
$\rightsquigarrow$ Idea:

Replace exponentiation on the group G by a group action of a group H on a set S:

$$H \times S \to S.$$

Square-and-multiply

Suppose $G \cong \mathbb{Z}/23$ and that Alice computes g^{13}.



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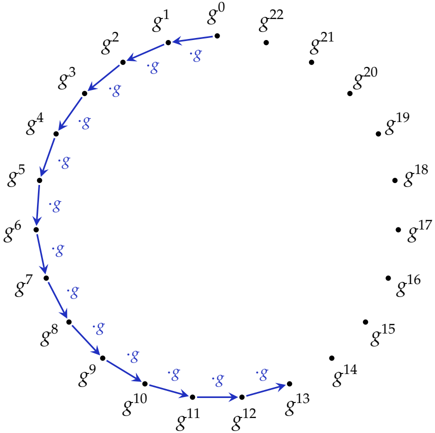

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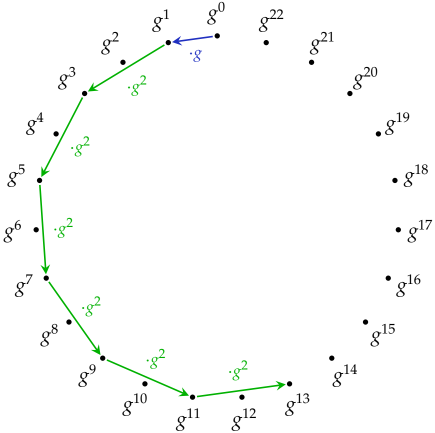

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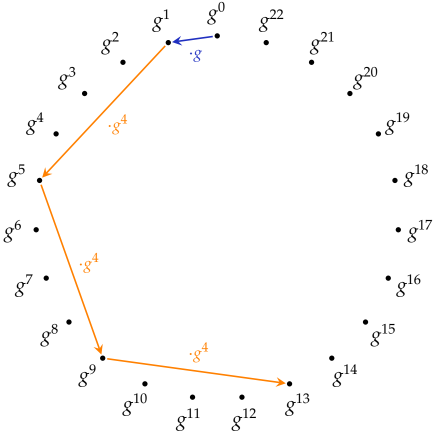

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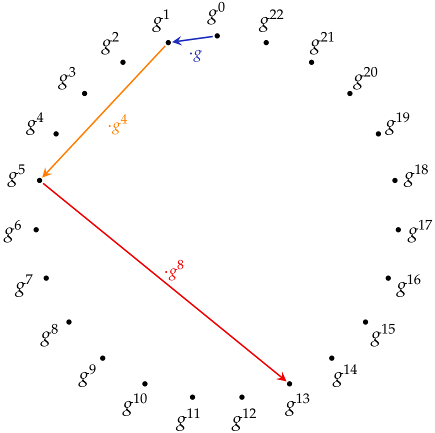

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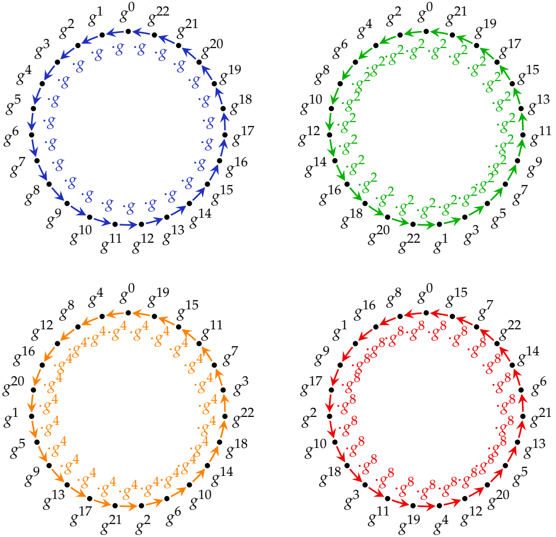

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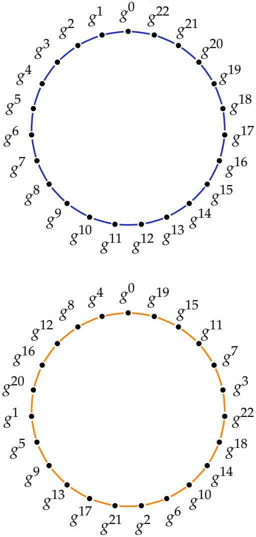

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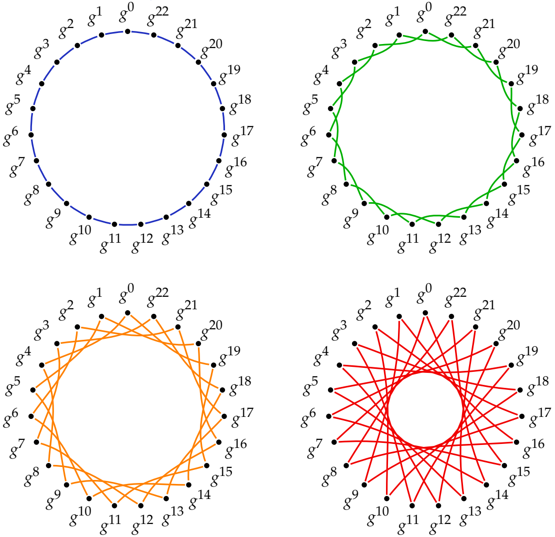

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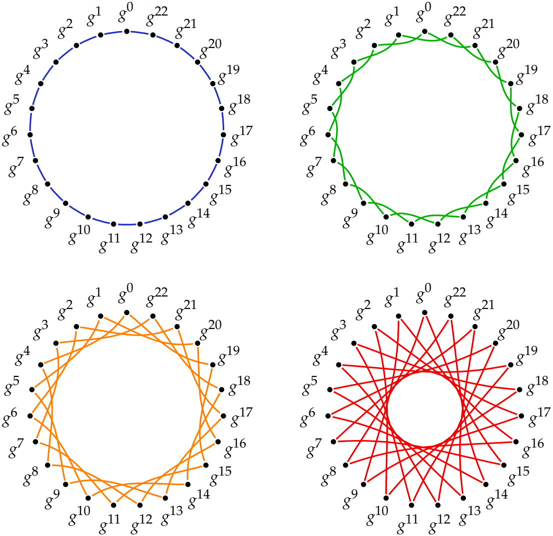
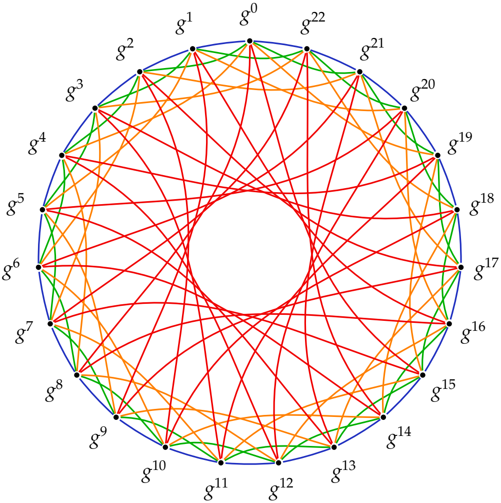


Cycles are compatible: [right, then left] = [left, then right], etc.

Union of cycles: rapid mixing



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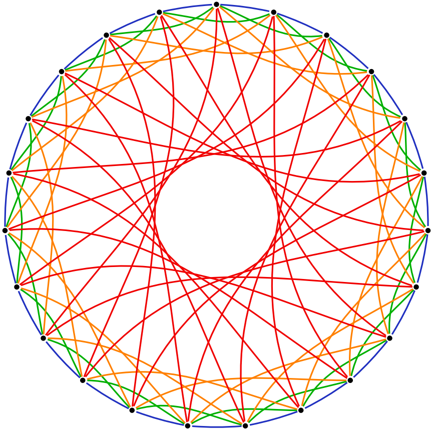
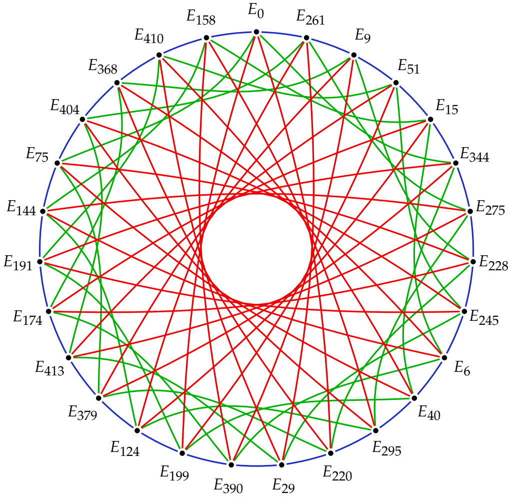


CSIDH: Nodes are now elliptic curves and edges are isogenies.

Graphs of elliptic curves



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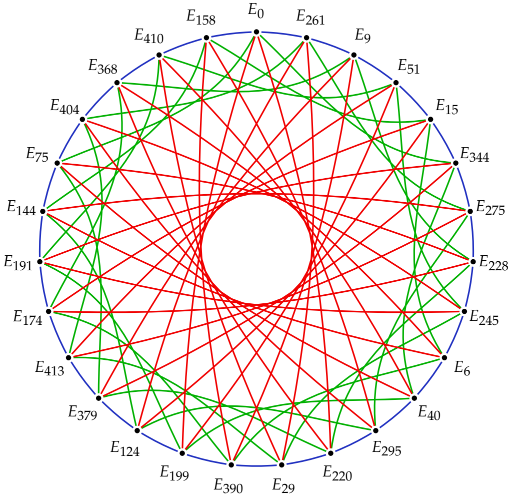

Nodes: Supersingular curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.

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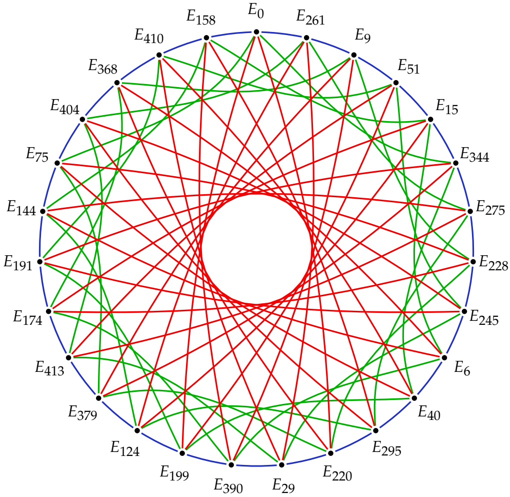


Nodes: Supersingular curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.
Edges: 3-, 5-, and 7-isogenies.

Quantumifying Exponentiation

- We want to replace the exponentiation map

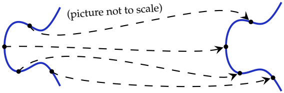
$$\begin{array}{ccc} \mathbb{Z} \times G & \to & G \\ (x, g) & \mapsto & g^x \end{array}$$

  by a group action on a set.

- Replace G by the set S of supersingular elliptic curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.
- Replace $\mathbb{Z}$ by a commutative group H... more details to come!
- The action of a well-chosen $h \in H$ on S moves the elliptic curves one step around one of the cycles.

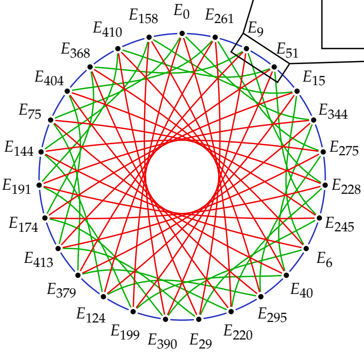
Graphs of elliptic curves

A 3-isogeny



$$E_{51}\colon y^2=x^3+51x^2+x \longrightarrow E_9\colon y^2=x^3+9x^2+x$$

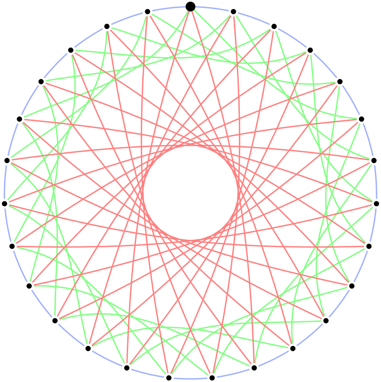
$$(x,y) \longmapsto \left(\frac{97x^3-183x^2+x}{x^2-183x+97},\ y\cdot\frac{133x^3+154x^2-5x+97}{-x^3+65x^2+128x-133}\right)$$



Diffie-Hellman on 'nice' graphs

Alice

$[+, -, +, -]$



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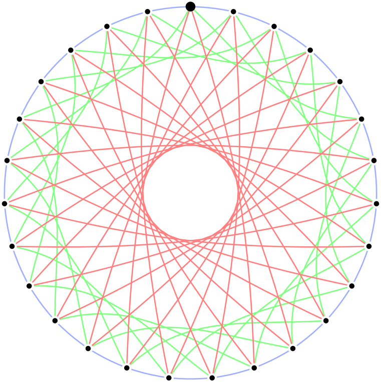

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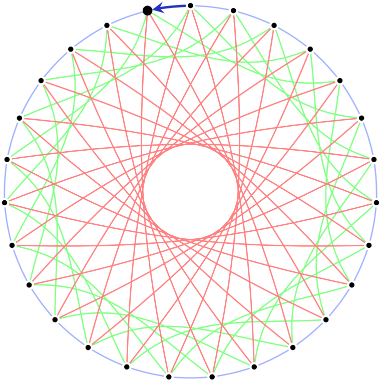

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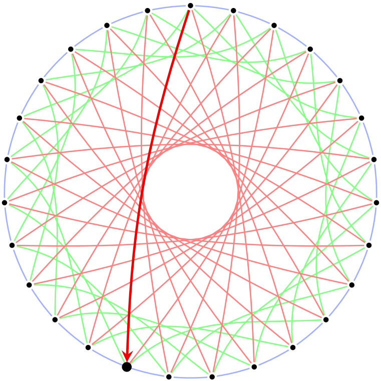

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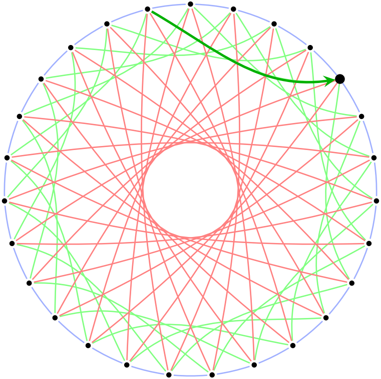

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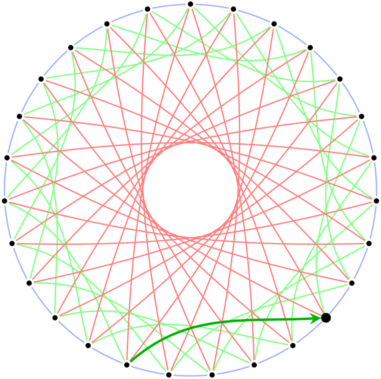

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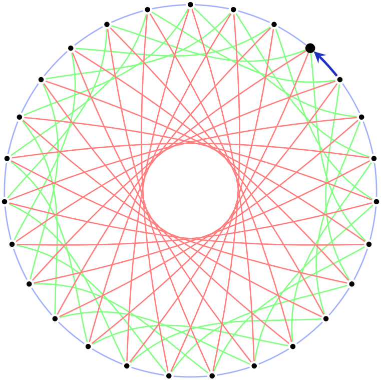

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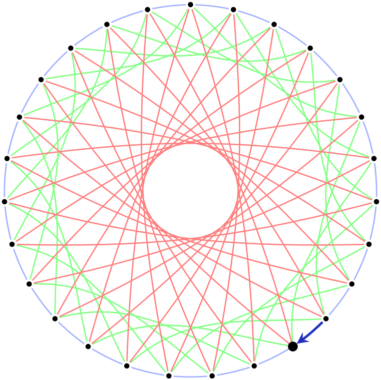

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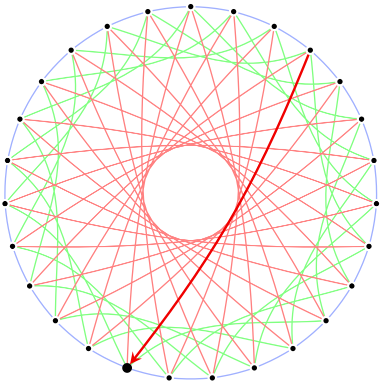

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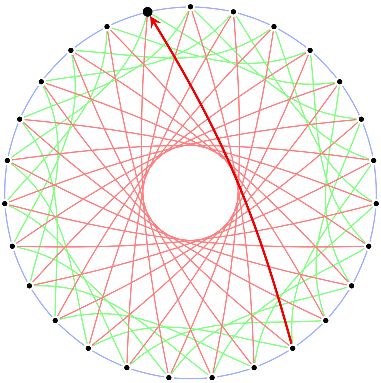

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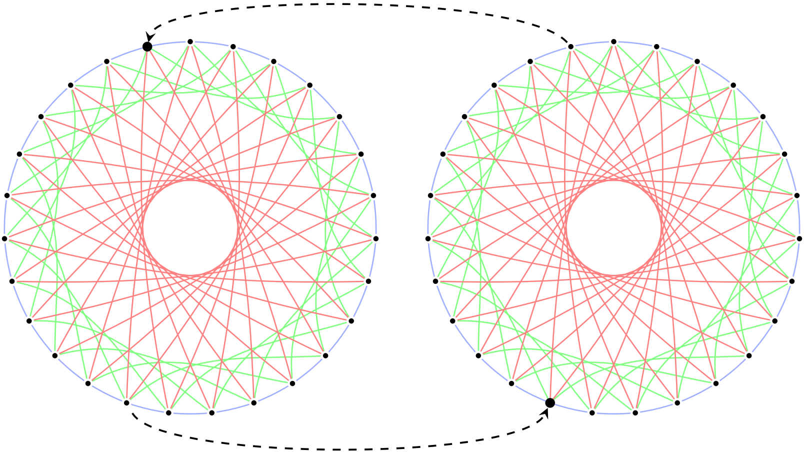

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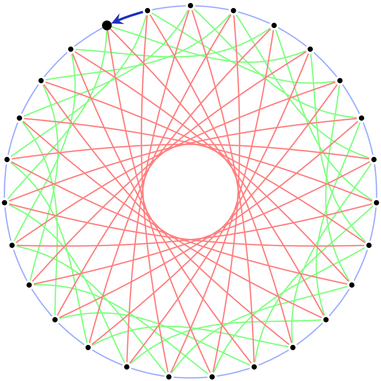

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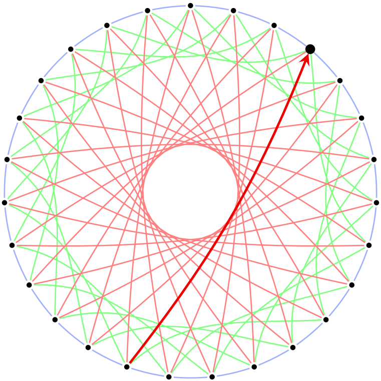

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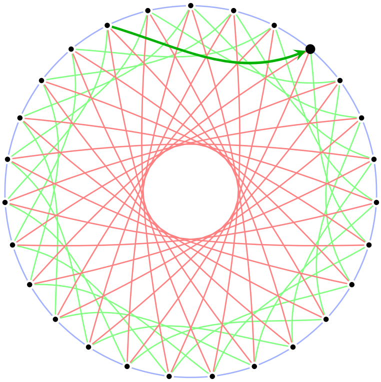

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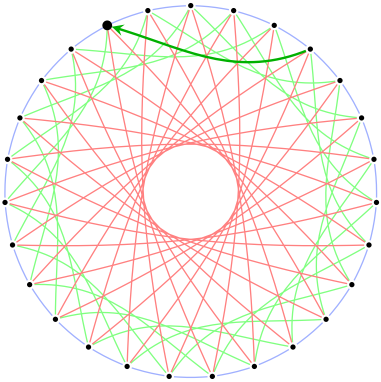

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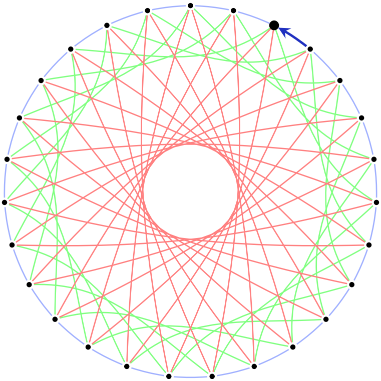

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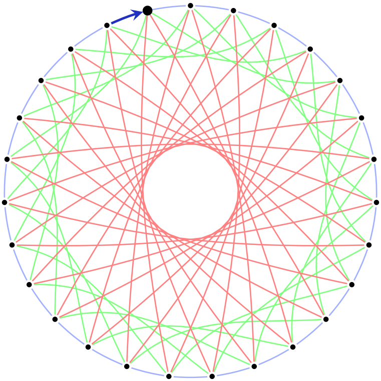

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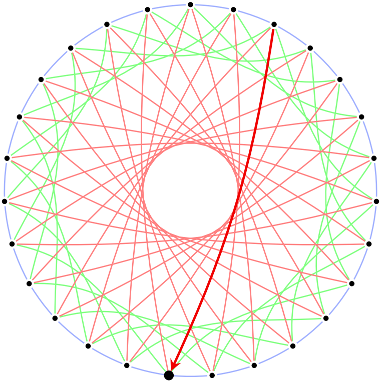

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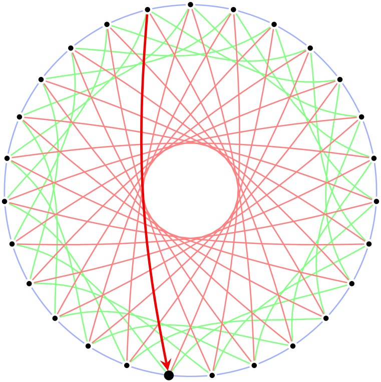

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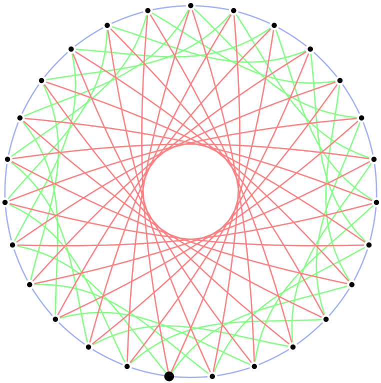

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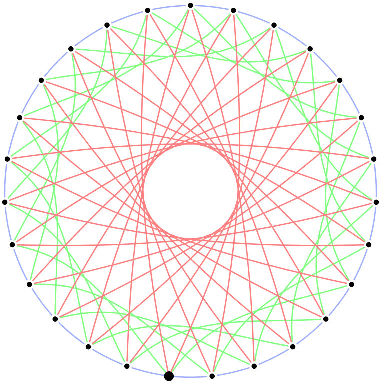

A walkable graph

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Important properties for such a walk:

IP1 ▸ The graph is a composition of compatible cycles.

IP2 ▸ We can compute neighbours in given directions.

Towards IP1: Isogeny graphs

First some reminders (see eg. autumn school slides):

- An elliptic curve $E/\mathbb{F}_p$ (for $p \geq 5$) is supersingular if $\#E(\mathbb{F}_p) = p + 1$.

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- For elliptic curves $E, E'/\mathbb{F}_p$ and a prime $\ell \neq p$, an ℓ-isogeny $f : E \to E'$ is an isogeny with $\# \ker(f) = \ell$.

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- If $f : E \to E'$ is an ℓ-isogeny, there is a unique dual isogeny $f^\vee : E' \to E$ such that $f^\vee \circ f = [\ell]$ is the multiplication-by-ℓ map on E.

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- The dual isogeny is also an ℓ-isogeny.

Towards IP1: Isogeny graphs

Definition

Let p and ℓ be distinct primes. The isogeny graph G_ℓ containing $E/\mathbb{F}_p$ is the graph with:

- Nodes: elliptic curves $E'/\mathbb{F}_p$ with $\#E(\mathbb{F}_p) = \#E'(\mathbb{F}_p)$ (up to $\mathbb{F}_p$-isomorphism).
- Edges: we draw an edge $E - E'$ to represent an ℓ-isogeny $f : E \to E'$ together with its dual ℓ-isogeny.

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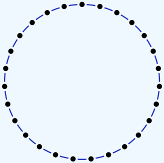
- In our example, these are

G_3:



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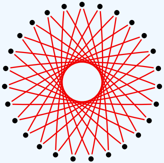
G_5:



Towards IP1: Isogeny graphs

Definition

Let p and ℓ be distinct primes. The isogeny graph G_ℓ containing $E/\mathbb{F}_p$ is the graph with:

- Nodes: elliptic curves $E'/\mathbb{F}_p$ with $\#E(\mathbb{F}_p) = \#E'(\mathbb{F}_p)$ (up to $\mathbb{F}_p$-isomorphism).
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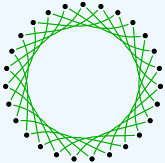

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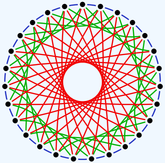

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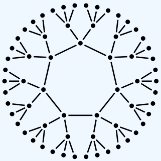
- Generally, the G_ℓ look something like

G_3:



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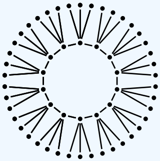

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- Equivalently: every node in G_ℓ should be distance zero from the cycle.
- Two nodes are at different distances from the cycle if and only if they have different endomorphism rings.

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Let $E/\mathbb{F}_p$ be an elliptic curve.

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- The Frobenius map

$$\begin{array}{rccc} \pi: & E & \to & E \\ & (x,y) & \mapsto & (x^p, y^p) \end{array}$$

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The $\mathbb{F}_p$-rational endomorphism ring $\mathrm{End}_{\mathbb{F}_p}(E)$ of an elliptic curve $E/\mathbb{F}_p$ is the set of $\mathbb{F}_p$-rational endomorphisms.

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Example

Let $p > 3$, let $E/\mathbb{F}_p : y^2 = x^3 + Ax^2 + x$ be a supersingular elliptic curve, and let π be the Frobenius endomorphism. Then

$$\pi \circ \pi = [-p]$$

and

$$\begin{array}{ccc} \mathbb{Z}[\sqrt{-p}] & \to & \mathrm{End}_{\mathbb{F}_p}(E) \\ n & \mapsto & [n] \\ \sqrt{-p} & \mapsto & \pi \end{array}$$

extends $\mathbb{Z}$-linearly to a ring homomorphism.

Towards IP1: Group action

For $p \equiv 3 \pmod 8$ and $p \geq 5$, if $E_A/\mathbb{F}_p : y^2 = x^3 + Ax^2 + x$ is supersingular, then $\mathrm{End}_{\mathbb{F}_p}(E_A) \cong \mathbb{Z}[\sqrt{-p}]$.

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- What is the action?

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- For $[I] \in \mathrm{Cl}(\mathbb{Z}[\sqrt{-p}])$, let $\tilde{I}$ be an integral representative of the ideal class $[I]$. Then
$$\begin{array}{ccc} \mathrm{Cl}(\mathbb{Z}[\sqrt{-p}]) \times S & \to & S \\ ([I], E) & \mapsto & f_{H_{\tilde{I}}}(E) \end{array}$$
is a free, transitive group action!

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$\rightsquigarrow$ there is a choice of $\ell_1, \ldots, \ell_n$ such that $G_{\ell_1} \cup \cdots \cup G_{\ell_n}$ is a composition of compatible cycles (IP1).

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- Choosing the direction in the graph corresponds to choosing this sign.

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- If $p \equiv -1 \pmod{\ell}$ then the action $[\langle \ell, \pi + 1 \rangle] * E$ gives an ℓ-isogeny in the '$-$' direction.

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Given the group action as above, Vélu's formulas give actual isogenies!
With our design choices all isogeny computations are over $\mathbb{F}_p$. [2]

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⇒ Tiny keys!

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- About $\sqrt{p}$ of all $A \in \mathbb{F}_p$ are valid keys.
- Public-key validation: Check that E_A has $p + 1$ points.

  Easy Monte-Carlo algorithm: Pick random P on E_A and check $[p + 1]P = \infty$.[3]

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- Say Alice's secret is isogeny is of degree $\ell_1^{e_1} \cdots \ell_n^{e_n}$. She knows the path, so can do only small degree isogeny computations, giving complexity $O(\sum e_i \ell_i)$. An attacker has to compute one isogeny of degree $\prod \ell_i^{e_i}$ (cf. isogeny evaluation complexity from David Jao's talk).
- Alternative way of thinking about it: Alice has to compute the isogeny corresponding to one path from E_0 to E_A, whereas an attacker has compute all the possible paths from E_0 to E_A.

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- Alternative way of thinking about it: Alice has to compute the isogeny corresponding to one path from E_0 to E_A, whereas an attacker has compute all the possible paths from E_0 to E_A.
- Best classical attacks are (variants of) meet-in-the-middle: Time $O(\sqrt[4]{p})$.

Quantum Security

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- Kuperberg later [Kup2] gave more trade-off options for quantum and classical memory vs. time.
- Childs-Jao-Soukharev [CJS] applied Kuperberg/Regev to CRS – their attack also applies to CSIDH.
- Part of CJS attack computes many paths in superposition.

Quantum Security

- The exact cost of the Kuperberg/Regev/CJS attack is subtle – it depends on:
  - Choice of time/memory trade-off (Regev/Kuperberg)
  - Quantum evaluation of isogenies

  (and much more).

[4] From [BLMP], using query count of [BS]. [BS] also study quantum evaluation of isogenies but their current preprint misses some costs.

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- For fastest variant of Kuperberg (uses billions of qubits), total cost of CSIDH-512 attack is about 2^{81} qubit operations.[4]

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Parameters

CSIDH-log p	intended NIST level	public key size	private key size	time (full exchange)	cycles (full exchange)	stack memory	classical security
CSIDH-512	1	64 b	32 b	85 ms	212e6	4368 b	128
CSIDH-1024	3	128 b	64 b				256
CSIDH-1792	5	224 b	112 b				448

Work in progress & future work

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- [Your paper here!]

Thank you!

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Mentioned in this talk:

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`https://quantum.isogeny.org`

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A faster way to the CSIDH
`https://ia.cr/2018/782`

Kup1 Kuperberg:
A subexponential-time quantum algorithm for the dihedral hidden subgroup problem
`https://arxiv.org/abs/quant-ph/0302112`

Kup2 Kuperberg:
Another subexponential-time quantum algorithm for the dihedral hidden subgroup problem
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Reg Regev:
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JLLR Jao, LeGrow, Leonardi, Ruiz-Lopez:
A polynomial quantum space attack on CRS and CSIDH
(MathCrypt 2018)

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